# Parallel and Distributed Algorithms

Winter 2009/2010

4

**Issue: 9.11.2009** **Due: 16.11.2009**

## Information

Solutions in english or german are fine.

## 4.1. Problem (10)

Assume that matrix multiplication of two $n \times n$ matrices runs in time $O(\frac{n^3}{p})$ on $p$ processors.

(a) Show how to compute the power $A^{2^k}$ of an $n \times n$ matrix $A$ in time $O(\frac{n^3}{p} \cdot k)$ with $p$ processors.

(b) Show how to compute the power $A^K$ of an $n \times n$ matrix $A$ in time $O(\frac{n^3}{p} \cdot \log_2 K)$ with $p$ processors.

(c) The sequence $(x_n \mid n \in \mathbb{N}_0)$ is described by the recurrence

$$x_0 = 1$$

$$x_m = a_1 \cdot x_{m-1} + a_2 \cdot x_{m-2} + \cdots + a_r \cdot x_{m-r}.$$

Show how to determine $x_n$ in time $O(\frac{r^3}{p} \cdot \log_2 n)$ with $p$ processors.
*Hint*: Construct a $r \times r$ matrix $M$ and a vector $v$ and show a connection between $x_n$ and $M^n \cdot v$

## 4.2. Problem (10)

In matrix multiplication entries are computed according to the formula $c_{i,j} = \sum_k a_{i,k} \cdot b_{k,j}$. In some applications similar expressions appear, but with addition and multiplication replaced by other operations.

(a) The transitive closure of a directed graph $G = (V, E)$ is the directed graph $G^* = (V, E^*)$, where we insert an edge $(i, j)$ into $E^*$ whenever there is a path in $G$ from $i$ to $j$. Show how to determine the transitive closure with the help of matrix multiplication.

Hint: Assume that $G$ has $n$ vertices and let $A$ be the adjacency matrix of $G$ with

$$A[i,j] := \begin{cases} 1 & (i,j) \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Interpret the power $A^{n-1}$ after replacing addition by $\vee$ and multiplication by $\wedge$.

(b) We are given $p$ processors. Implement your solutions for (a), assuming that $p$ processors are available. Determine running time and efficiency, if the sequential time complexity is $\Theta(n^3)$ for graphs with $n$ vertices.

### 4.3. Problem (10)

In Problem 4.2 we discussed how to use matrix multiplication in order to determine the transitive closure of a directed graph. Here we discuss the shortest path problem: we are given a directed graph $G$ with $n$ vertices and non-negative weights $w_{i,j}$ for its edges $(i,j)$. We have to determine the length of the shortest path for any pair of vertices.

We define the matrix

$$A[i,j] := \begin{cases} w_{i,j} & (i,j) \text{ is an edge of } G, \\ \infty & \text{otherwise.} \end{cases}$$

Which operations should replace addition and multiplication, so that $A^{n-1}[i,j] =$ the length of a shortest path from $i$ to $j$?